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# SELF-CHANNELLING OF SURFACE WATER WAVES IN THE PRESENCE OF AN ADDITIONAL SURFACE PRESSURE

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(Received 15 August 1998, revised and accepted 3 February 1999)

**Abstract** – Nonlinear instability of long periodic waves of a small amplitude on a surface of a water layer of finite depth either subjected to surface tension or in the presence of an elastic ice-sheet floating on the water surface is treated. Wave processes in both cases are assumed to be described by a model equation which generalizes the Kadomtsev-Petviashvili equation to the presence of higher order dispersive effects. The treatment is based on the analysis of the Benjamin-Feir type instability, governed by the Davey-Stewartson equations for slowly varying in time and space complex amplitudes of periodic waves. The homogeneous periodic wave is shown to be unstable under perturbations, transversal to a direction of wave propagation. Such kind of instability leads to a formation of a lattice of essential wave-guides, i.e. waves, periodic in the direction of propagation and localized in the transversal direction. Some natural effects of ice damage, which can be explained with the help of such an instability, are discussed.© Elsevier, Paris

## 1. Introduction

Wave propagation on a water surface in the presence of higher dispersive effects, caused either by surface tension or by an elastic ice-sheet drew considerable attention in the literature. The problem is attractive both as a theoretical subject as well as due to its importance to applications. From the theoretical point of view models of gravity-capillary and water-ice (flexural-gravity) waves allow application of refined mathematical tools for studying wave phenomena (see e.g., [6–8]). Unlike the gravity-capillary case, where comparison of theoretical results, obtained from the non-viscous description, and experimental data is hard to perform (see e.g., [1]), flexural-gravity waves have large practical significance. From the engineering standpoint there are questions of stress control in the ice cover in the neighbourhoods of facilities built upon ice, break-up of ice in off-shore regions, performance of ice-breakers, etc.

One of the first questions with regard to any water model is that of its stability, in particular, of the stability of periodic wave patterns taking place in the model. Unstable periodic waves can not represent a physical end state. Linear stability analysis for such waves yields exponentially growing with time perturbations. Though, taking into account a nonlinearity, may lead to a formation of new bounded wave patterns, which are different from the initial periodic wave.

The equation considered here

$$\partial_{tx}\eta + \partial_x(\eta\partial_x\eta) + s\partial_x^4\eta + \partial_x^6\eta + \partial_y^2\eta = 0, \quad s = \pm 1, \tag{1.1}$$

where $t$, $x$, $y$, are dimensionless temporal and spatial variables, correspondingly, $\eta$ - dimensionless surface deviation, is a generalisation of the Kadomtsev-Petviashvili (KP) equation. The equation (1.1) is derived in [5] from the full 3D system of Euler's equations for long gravity-capillary waves of small amplitude when the

- Supported by a Research Fellowship from the Alexander von Humboldt Foundation.

dimensionless Bond number $b$ is close to $1/3$, and also for surface water waves in the presence of an elastic ice-plate. In both cases a liquid of finite depth is considered. For large enough surface tension ($b > 1/3$), $s = -1$. For $b < 1/3$ and in flexural-gravity case $s = 1$. The deflection of the elastic ice-sheet, floating on the water surface, is governed by the equations of the theory of thin plates, namely, the Kirchoff-Love elastic plate model is used for ice. The experiments, reported in [10], show that the ice sheet exhibits an elastic behaviour for a wide set of physically relevant conditions.

We examine here the nonlinear instability of homogeneous periodic wave train with respect to perturbations which are transversal to a direction of propagation. Such kind of instability is called self-channelling (see e.g., [9]), and leads to a formation of essential wave-guides, periodic in the direction of propagation of the initial wave and localized in the transversal direction.

We proceed as follows. In section 2 we derive from (1.1) the Davey-Stewartson equations for amplitude-envelopes and show that these equations yield the instability of homogeneous periodic wave patterns, leading to the self-channelling. Section 3 contains our conclusion and a discussion of effects of break-up of shore-fast ice. We explain this phenomenon by self-channelling under the ice cover of incident periodic waves. Some possible extensions of instability analysis given here are also discussed. In Appendix the subcritical wave-guide-type solution of (1.1), indicating the presence of the instability in question, is obtained. To get this solution, the centre-manifold reduction of the dynamical system, describing the travelling wave-type solutions of (1.1) is used. The flow on the centre-manifold is then put in normal form. The normal form system corresponds to the simple bifurcation from the quiescent state, and can be integrated explicitly to any algebraic order with respect to the small parameter (wave amplitude). The persistence for the full system of the solutions of the normal form system in the case in question is a known fact (see e.g., [8]).

## 2. The Davey-Stewartson equations. Self-channelling of periodic waves.

### 2.1. *Derivation of the Davey-Stewartson equations. Wave-guide solutions.*

We proceed by seeking asymptotic solution of (1.1) of the form

$$\eta = \epsilon A(T,X,Y)\exp i\theta + \epsilon^2 A_2(T,X,Y)\exp 2i\theta + \text{c.c} + \epsilon^2 A_0(T,X,Y) + O(\epsilon^3), \tag{2.1}$$

where

$$\theta = k(x - Vt), \quad V = V_1 - \epsilon^2, \quad T = \epsilon t \quad X = \epsilon x, \quad Y = \epsilon y,$$

$\epsilon$ is a small parameter, $A$ and $A_2$ are slowly varying complex amplitudes, $A_0$ is a real function, corresponding to a mean flow and $V_1$ is the critical speed. Next we collect the terms proportional to $\exp(i\theta)$. At leading order in $\epsilon$ the dispersion equation $V_1 = -k^2 + k^4$ is satisfied. At the order $\epsilon^2$ one gets

$$\frac{\partial A}{\partial T} + \omega' \frac{\partial A}{\partial X} = O(\epsilon), \tag{2.2}$$

where $\omega = V_1 k$, and prime denotes the differentiation with respect to the wavenumber $k$. The equality (2.2) suggests the change of variables $X \to X - \omega'(k)T$, $\tau = \epsilon T$, which we use later on, preserving notation $X$ for the combination $X - \omega'(k)T$. At the order $\epsilon^3$ we get

$$ik\frac{\partial A}{\partial \tau} - k^2 A + (-6k^2 + 15k^4)\frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial X \partial T} - k^2(A_2 A^* + A A_0) + \frac{\partial^2 A}{\partial Y^2} = 0, \tag{2.3}$$

where asterisk denotes complex conjugation. Collecting the terms at $\epsilon^4$ leads to

$$\frac{\partial^2 A_0}{\partial X \partial T} + \frac{\partial^2}{\partial X^2}|A|^2 + \frac{\partial^2 A_0}{\partial Y^2} = 0. \tag{2.4}$$

The equation (2.4) to the required order is equivalent to

$$-\omega' \frac{\partial^2 A_0}{\partial X^2} + \frac{\partial^2}{\partial X^2}|A|^2 + \frac{\partial^2 A_0}{\partial Y^2} = 0,$$

as

$$\frac{\partial^2 A_0}{\partial X \partial T} = -\omega' \frac{\partial^2 A_0}{\partial X^2} + O(\epsilon).$$

Collecting the terms proportional to $\epsilon^2 \exp(2i\theta)$ one gets

$$A_2 = \frac{A^2 \Delta}{k^2}, \tag{2.5}$$

where $\Delta < 0$ for $k > 1/\sqrt{5}$ is given in Appendix. Finally, substituting (2.5) into (2.3) we arrive at the Davey-Stewartson equations

$$\begin{aligned} &iA_\tau - kA + \frac{\omega''}{2} A_{XX} - \frac{\Delta}{k} A|A|^2 - kAA_0 + \frac{1}{k} A_{YY} = 0 \\ &-\omega'(k) A_{0XX} + |A|^2_{XX} + A_{0YY} = 0, \end{aligned} \tag{2.6}$$

where subscripts $\tau$, $X$ and $Y$ denote differentiation with respect to the corresponding variables.

The equations (2.6) have the particular solution $A = \Psi(Y) \exp i\nu X$, $A_0 = 0$, where $\Psi(Y)$ is a real function, satisfying

$$\ddot{\Psi}(Y) = \Delta_1 \Psi(Y) + \Delta \Psi^3(Y), \quad \Delta_1 = k^2 + \frac{\omega'' k}{2} \nu^2, \tag{2.7}$$

where dots denote differentiation with respect to $Y$. The localized solution of (2.7) is given by

$$\Psi(Y) = \pm \sqrt{\frac{-2\Delta_1}{\Delta}} \operatorname{sech} \sqrt{\Delta_1} Y. \tag{2.8}$$

For any $\nu$ this solution requires $\Delta_1 > 0$, or $k > \sqrt{3/10} > 1/\sqrt{5}$. The parameter $\nu$ defines an arbitrary amplitude of (2.8). Comparison of (A.9), (A.10) with (2.7), (2.8) yields for $\nu = O(\epsilon)$

$$\Psi(Y) = \frac{\widehat{a}_0}{2\epsilon}, \quad \epsilon^2 = -\mu.$$

## 2.2. *Instability analysis*

Introducing the new real variables given by $A = a \exp i\psi$ one obtains from (2.6)

$$\begin{aligned} &a_\tau + \frac{\omega''}{2}(2a_X\psi_X + a\psi_{XX}) + \frac{1}{k}(2a_Y\psi_Y + a\psi_{YY}) = 0 \\ &-a\psi_\tau - ka + \frac{\omega''}{2}(a_{XX} - a\psi_X^2) - \frac{\Delta}{k}a^3 - kA_0 a + \frac{1}{k}(a_{YY} - a\psi_Y^2) = 0 \\ &-\omega' A_{0XX} + (a^2)_{XX} + A_{0YY} = 0. \end{aligned} \tag{2.9}$$

The solution of (2.9) corresponding to a plane periodic wave, has the form

$$a = a^0 = \text{const}, \quad \psi = \psi^0 = (-k - (a^0)^2 \frac{\Delta}{k})\tau, \quad A_0 = 0.$$

As a terminology, we call this solution homogeneous periodic wave, because it does not depend on the transversal coordinate $Y$. Substituting

$$a = a^0 + \delta a, \quad \psi = \psi^0 + \delta\psi, \quad A_0 = \delta A_0,$$

into (2.9) and assuming

$$\delta a = \alpha_1 \exp i(\kappa_{\|} X + \kappa_{\perp} Y - \Omega\tau), \quad \delta\psi = \alpha_2 \exp i(\kappa_{\|} X + \kappa_{\perp} Y - \Omega\tau),$$
$$\delta A_0 = \alpha_3 \exp i(\kappa_{\|} X + \kappa_{\perp} Y - \Omega\tau),$$

where $\alpha_i$, $i$=1,2,3 are constants, we get the dispersion relation for (2.9)

$$\Omega^2 = \frac{1}{4}\left(\omega''\kappa_{\|}^2 + 2\frac{\kappa_{\perp}^2}{k}\right)^2 + (a^0)^2\left(\frac{k\kappa_{\|}^2}{\omega'\kappa_{\|}^2 - \kappa_{\perp}^2} + \frac{\Delta}{k}\right)\left(\omega''\kappa_{\|}^2 + 2\frac{\kappa_{\perp}^2}{k}\right). \tag{2.10}$$

First we note, that the equation (2.10) contains the self-modulational instability, which is governed by the "embedded" in (1.1) Kawahara equation for plane waves

$$\partial_t\eta + \eta\partial_x\eta + \partial_x^3\eta + \partial_x^5\eta = 0.$$

This instability leads to a decay of periodic wave into a sequence of envelope solitary waves for $k \in (\sqrt{3/10}, \sqrt{3/5})$ [2]. It means, that periodic waves considered here, for a given range of wave lengths are subjected to the self-modulation with respect to longitudinal perturbations, which are parallel to the direction of propagation of such a wave ($\kappa_{\perp} = 0$).

In the present paper we focus our attention on homogeneous transverse perturbations, i.e when $\kappa_{\|} = 0$, and perturbations do not depend on the $X$ coordinate. The dispersion equation (2.10) then takes the form

$$\Omega^2 = \frac{\kappa_{\perp}^4}{k^2} + 2\frac{(a^0)^2}{k^2}\Delta\kappa_{\perp}^2. \tag{2.11}$$

Wave numbers $\kappa_{\perp}$, obeying (2.11) and lying inside the segment $[\kappa_{max}, \kappa_0)$, where $\kappa_{max}^2 = -(a^0)^2\Delta$ and $\kappa_0^2 = -2(a^0)^2\Delta$, $\kappa_{\max} < \kappa_0$, correspond to exponentially growing with time perturbations. The value of $\kappa_0$ gives the threshold of instability, i.e. for all $\kappa_{\perp} \geq \kappa_0$, the periodic wave in question is stable. From (2.11) one gets the maximal growth rate of perturbations: $\Omega(\kappa_{max}) = -(a^0)^2\Delta/k$.

For perturbations, depending only on $Y$, assuming

$$\delta a(0, Y) = a^0\widehat{A}(Y), \quad \delta\psi(0, Y) = 0,$$

where, without any loss of generality $\widehat{A}(Y)$ is set to be even, we get the initial value problem

$$\delta a_{\tau\tau} - 2\frac{(a^0)^2}{k^2}\Delta\delta a_{YY} + \frac{1}{k^2}\delta a_{YYYY} = 0$$
$$\delta a(0, Y) = a^0\widehat{A}(Y), \quad \delta a_\tau(0, Y) = 0. \tag{2.12}$$

The initial value problem (2.12) can be easily solved. Its solution is given by

$$\delta a(\tau, Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\kappa)\cos[\kappa Y - \Omega(\kappa)\tau]\, d\kappa, \tag{2.13}$$

where

$$R(\kappa) = 2a^0 \int_0^\infty \widehat{A}(Y) \cos \kappa Y \, dY, \quad \Omega(\kappa) = \frac{|\kappa|}{k} \sqrt{\kappa^2 - \kappa_0^2}.$$

To analyse the asymptotic behaviour of (2.13) for large time we follow [9]. This analysis, in fact, describes the linear stage of evolution of perturbations, and therefore we are restricted to a bounded value of time. Yet, this value may be taken large enough, if we deal with small initial perturbations $\widehat{A}(Y)$. In our analysis we distinguish the domains of different behaviour of (2.13): $Y << \kappa_0\tau$ and $Y >> \kappa_0\tau$. In other words, we consider the evolution of perturbations in the vicinity of the origin $Y = 0$ and far from the origin. In the first domain, using the method of steepest descent, one finds that for large $\tau$ the main contribution to the integral (2.13) is given by the mode, determined by the condition $d\Omega/d\kappa = 0$. This equation has the root given by $\kappa = \kappa_{max}$, and the dominant term of the asymptotic reads

$$\delta a(\tau, Y) \sim C \exp[\kappa_{max}\tau] \cos\left(\frac{\kappa_0 Y}{\sqrt{2}} + \lambda\right),$$

with constants $C$ and $\lambda$ to be determined from the Cauchy data. Therefore, in the first domain we have the exponential growth of perturbations with the maximum growth rate $\kappa_{max}$. The group speed of these perturbations equals to zero.

At large values of $|Y/(\kappa_0\tau)|$ an asymptotic expression for (2.13) may be found with the help of the stationary phase method. The wave numbers for dominant modes, then, are given by $d\Omega(\kappa)/d\kappa = |Y/\tau|$. This equation has the roots

$$\kappa_1 = \frac{k}{2}\left|\frac{Y}{\tau}\right| > \kappa_0, \quad \kappa_2 = 1 + \frac{1}{2}\frac{\kappa_0^2\tau^2}{k^2Y^2} > \kappa_0.$$

Therefore, for large $\tau$, (2.13) is given by the combination of two modes. These waves are stable ($\kappa_{1,2} > \kappa_0$), and propagate with the nonzero group speed.

Thus the scenario of the evolution of perturbations at the initial stage is as follows (for general description see e.g., sec 28, 29 in [9]). In the central region we have the large scale modulations with wave numbers lying in a small neighbourhood of $\kappa_{max}$. These waves are close to steady waves with zero group velocity. For increasing $\tau$, the linear approximation is no more valid, and the exponential growth of amplitudes is damped by nonlinearity. The depth of modulations increases with time and the contribution of nonlinearity forces the modulated wave to decay into localized waves, described by (2.8). The width of the region of instability in question enlarges with $\tau$, as the modulation wave propagates from the boundary of this region with speeds, growing for increasing $Y$. For large enough absolute values of $Y$ (in the region $|Y| >> \kappa_0\tau$) one has the stable wave packets, with wavenumbers $\kappa_{1,2}$ greater than the value of the threshold of instability $\kappa_0$. These waves propagate away from the origin with the group speed which is equal to $|Y/\tau|$.

The development of instability of homogeneous in the $y$-direction periodic waves, subject to transverse perturbations implies also some peculiarities of evolution of weakly non-homogeneous initial disturbances. As an example, we consider here the initial disturbance of the form

$$A(t = 0, y) = a^0 f\left(\frac{\epsilon y}{L}\right), \quad f(\xi) \to 0, \quad \text{as } |\xi| \to 0, \tag{2.14}$$

where $f(\xi)$ is a dimensionless function, normalised such that $\max f = 1$. If the width of this wave is much greater than the width of the corresponding solitary wave (2.8), i.e $L >> l$, then (2.14) has to decay with time into a sequence of parallel solitary waves, which form are similar to that one, given by (2.8). Thus, the initial surface wave, which is periodic in $x$, and depends on $y$ via (2.14), decays into several parallel "channels".

The surface deviation for each "channel" is determined from (2.8) and (2.1). The shape of the "channel" is qualitatively similar to that one presented in Fig.3 in Appendix.

## 3. Conclusion and discussion

The outcome of the analysis presented here concerns the instability properties of periodic waves on a water surface in the presence of additional surface pressure. This pressure is caused either by capillary effects or by an elastic ice-sheet floating on the surface of a water layer. The waves under consideration are assumed to be of a small amplitude and to have a size, which considerably exceeds the depth of the water layer. More precisely, let us introduce two small parameters in the problem governed by full Euler's equations: $\varepsilon_1 = \eta_0/H$ and $\varepsilon_2 = H^2/\Lambda^2$, where $\eta_0$ is a characteristic wave amplitude, $H$ - the depth of the water layer and $\Lambda$ is a characteristic wave length in the $x$-direction. Assuming $\varepsilon_1 = O(\varepsilon_2)$ for water waves beneath the elastic ice sheet and $\varepsilon_1 = O(\varepsilon_2^2)$ for gravity-capillary waves with the Bond number close to 1/3, one obtains, that the surface deviation $\eta$ obeys the equation (1.1) for waves, propagating in one direction (for details see [5]). The above mentioned surface effects cause the additional dispersion, which is given by the term with the sixth derivative in (1.1).

In this work we focus our attention on the instability of waves, periodic in the direction of propagation and either homogeneous or having large enough support in the transversal direction. We consider the instability with respect to a particular type of perturbations, which are transversal to the direction of propagation along the $x$-axis and homogeneous in $x$. The periodic waves subjected to such a kind of perturbation is found to decay into a sequence of parallel wave guides or "channels". Each of these wave guides represents the wave, propagating in the $x$-axis direction and localized in the transversal direction (the $y$-axis direction).

It was observed in reported in [11] experiments, performed in St. Anthony Bight, Newfoundland, that incoming sea waves can cause break-up of seemingly robust fast ice, which is destroyed in just a few hours by waves incident on the ice edge from open sea. In [3] it was shown, that the pressure and bending momentum achieve their maximal values on crests of waves, propagating under the elastic ice-sheet. In practice, ice is expected to be cracked near the points of high pressure formation. Figure 1 in [11] shows, that cracking of shore-fast ice occurs at uniform intervals, resulting in the formation of floating strips of approximately equal width. This regular structure of shore-fast ice break-up can be explained by self-channelling of incident from clean water, normally to the ice edge, periodic wave. When such a wave comes under the ice cover, it can decay, due to self-channelling, into a sequence of parallel wave guides, having the same direction of propagation as the incident wave. Localisation of these wave guides has a regular character, i.e. their crests are situated at an approximately equal distance from each other. The maximal pressure, formed at crests of such a lattice of wave-guides, can result in the appearance of parallel cracks in ice, which are situated at uniform intervals in the direction transversal to that one of wave propagation.

The presence of self-channelling in the wave processes governed by equation (1.1) is indicated by existence of the wave guide solution (A.11) of (1.1). This solution represents a subcritical travelling wave having the greater amplitude for the smaller speed. This property of the wave in question suggests the following physical mechanism of the development of instability of a homogeneous wave. If the amplitude of a linear wave increases in some point, it causes the decrease of the speed of the wave in this point, which, in its turn, provokes the further increase of the amplitude. The process is repeated until the growth of the amplitude is compensated by dispersive and nonlinear effects. The final stage of evolution of the initially homogeneous wave is given by waves, belonging to the wave family, parametrised by $\nu$

$$\eta = 2\epsilon\Psi(Y)\mathcal{R}e \exp i\nu X \exp ik(x - Vt) + O(\mu), \quad V = V_1 + \mu, \quad \epsilon^2 = -\mu, \quad \mu < 0 \tag{3.1}$$

where $\Psi(Y)$ is determined via (2.8). The wave (A.11) is a member of (3.1) for $\nu = 0$.

We have considered a particular type of perturbations which propagate along the $y$ axis direction. There are a lot of other unstable directions "hidden" in the dispersion equation (2.10). Among them one finds the $x$-axis direction. This fact makes it possible to suppose, that the wave-guide solution can be destroyed by perturbations, propagating along the $x$-axis, and that the equation (2.6) may admit the travelling waves, totally localised in space. These waves are expected to represent the final stage of evolution of periodic wave under nonhomogeneous oblique perturbations. The instability analysis in this case is more involved, and is planned to be a subject of the forthcoming investigation.

## Appendix A. Wave-guide solution of (1.1)

We consider travelling wave solutions of (1.1), propagating with a speed $V$ in the $x$-direction, having the form $\eta = \eta(x - Vt, y)$. These solutions satisfy

$$\partial_{yy}\eta - V\partial_{xx}\eta + \partial_x(\eta\partial_x\eta) + s\partial_x^4\eta + \partial_x^6\eta = 0 \tag{A.1}$$

where the former notation $x$ for the combination $x - Vt$ is preserved. Assume $V = V_1 + \mu$ where $V_1$ is some fixed value of the speed to be specified below, and $\mu$ is a small parameter. We choose $y$ as an unbounded dynamical variable. Consequently, the equation (A.1) can be written in the form of the dynamical system

$$\begin{aligned} \dot\eta &= \eta_1 \\ \dot\eta_1 &= V_1\partial_x^2\eta - s\partial_x^4\eta - \partial_x^6\eta + \mu\partial_x^2\eta - (\partial_x\eta)^2 - \eta\partial_x^2\eta, \end{aligned} \tag{A.2}$$

where dot denotes differentiation with respect to $y$. Then (A.2) may be represented as follows

$$\dot{\mathbf{w}} = \mathcal{A}\mathbf{w} + \mathcal{F}(\mu, \mathbf{w}), \tag{A.3}$$

with

$$\mathbf{w} = \{\eta, \eta_1\}^t, \quad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ V_1\partial_x^2 - s\partial_x^4 - \partial_x^6 & 0 \end{pmatrix}, \quad \mathcal{F}(\mu, \mathbf{w}) = \{0,\ \mu\partial_x^2\eta - (\partial_x\eta)^2 - \eta\partial_x^2\eta\}^t. \tag{A.4}$$

Here $\mathcal{A}$ is the linearised about the quiescent state right hand side of (A.2), and $\mathcal{F}$ gives the nonlinear terms. The system (A.3)-(A.4) is a reversible one, i.e. $\mathcal{AR} = -\mathcal{RA}$, $\mathcal{F}(\mu, \mathcal{R}\mathbf{w}) = -\mathcal{RF}(\mu, \mathbf{w})$, where $\mathcal{R} = diag\{1, -1\}$. Note, that the system (A.3)-(A.4) has also two extra symmetries: translational $\tau_a\mathbf{w}(x, \cdot) = \mathbf{w}(x + a, \cdot)$ and reflectional $\mathcal{S}\mathbf{w}(x, \cdot) = \mathbf{w}(-x, \cdot)$. Yet, we need only reversibility for our analysis. We seek solutions of (A.1) which are $2l$-periodic in the $x$-direction and even in $x$. To obtain the dispersion relation we make use of the substitution $\eta = \exp[\sigma y]\cos(nkx)$, $k = \pi/l$, $n \in \mathbb{N}^+$, $\sigma \in \mathbb{C}$. Then the spectrum of the operator $\mathcal{A}$ consists of eigenvalues satisfying

$$\sigma^2 = -Vn^2k^2 - sn^4k^4 + n^6k^6. \tag{A.5}$$

Due to the reversibility and the fact that the operator $\mathcal{A}$ is real, the eigenvalues are symmetric about real and imaginary axis on the complex $\sigma$-plane.

A bifurcation occurs when eigenvalues of $\mathcal{A}$ arrive at the imaginary axis in pairs. Therefore we put $\sigma = i\,q$. By the argument above it suffices to consider $q \geq 0$. Then (A.5) takes the form

$$f(q) = q^2 = Vn^2k^2 + sn^4k^4 - n^6k^6 = g(n). \tag{A.6}$$

The eigenvalues are coming to the imaginary axis when the graph $f(q)$ touches the horizontal line $g(n)$, $n \in \mathbb{N}^+$ (see Fig.1). The first bifurcation takes place when the line $g(1)$ touches the graph $f(q)$ at $q = 0$. Such kind of bifurcations was investigated previously in [4] for the KPI equation, i.e. for $s = -1$. For KPII ($s = 1$) this bifurcation does not take place, because all $g(n)$ are greater than zero for any speed $V$ and, consequently there are infinitely many imaginary eigenvalues. But in our case of the generalized KP equation the bifurcation is

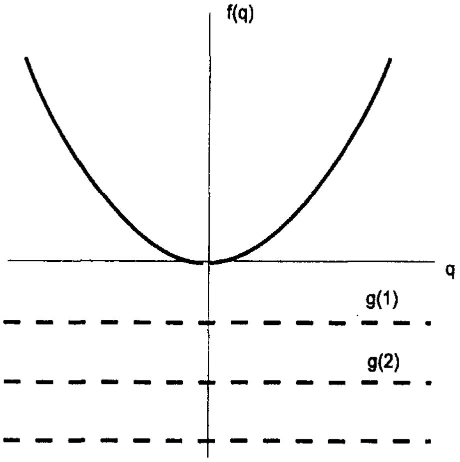


FIGURE 1. The comparative location of the polynomial $f(q)$ and the levels $g(n)$ for $s = 1$, $V < V_1$.

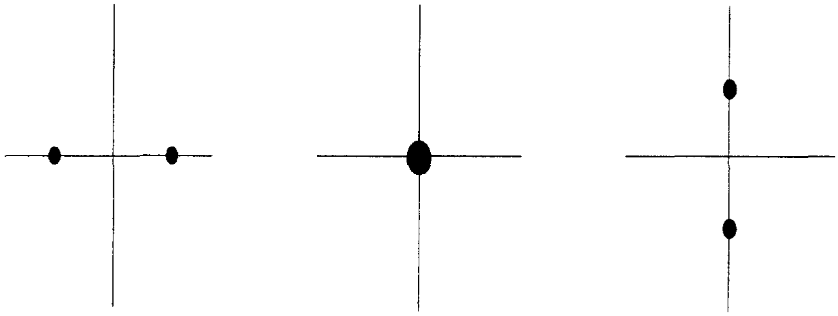


FIGURE 2. Dynamics of critical eigenvalues for $V = V_1 + \mu$, additional double zero eigenvalue is omitted.

possible for $s = 1$ as well. Further we restrict our analysis to the case $s = 1$, which covers also the flexural-gravity waves. For $s = 1$ the bifurcation in question takes place if $k > 1/\sqrt{5}$. It is also easy to verify that for such $k$ $g(n) > g(n+1)$. From (A.6) we deduce that the first bifurcation occurs when $V = V_1 = -k^2 + k^4$. Note that a new pair of eigenvalues comes to the imaginary axis (i.e. a new bifurcation takes place) when the graph of the polynomial $f(q)$ touches the next "level" $g(n)$, parallel to the $q$-axis. The eigenvalues then diverge along the imaginary axis. The multiplicity of the moving eigenvalues is one, due to the fact that we consider only even in $x$ solutions. The dynamics of eigenvalues coming for the first time to the imaginary axis is shown in Fig.2

**Remark A.1.** *Note, that zero is an eigenvalue for any speed $V$, due to the invariance $\eta \to \eta + const$, $V \to V + const$, and it is double due to reversibility. This zero eigenvalue increases the dimension of the centre manifold by 2. However, both these 2 additional dimensions can be eliminated by using the identities*

$$\int_{-l}^{l} \eta_1 \, dx = 0, \quad \int_{-l}^{l} \eta \, dx = const,$$

*where the constant in the last equality is set to 0, which is the case for solitary waves. These equalities are obtained by integrating (A.1) with respect to x over the period. For details see [5].*

Next define the spaces

$$\mathcal{X}_l = H_e^3 \times H_e^0, \quad \mathcal{Y}_l = H_e^6 \times H_e^3,$$

where $H_e^m$ denotes the Sobolev spaces of $2l$-periodic and even functions. We claim also that our solutions should satisfy the indentities in the Remark.

The linear operator $\mathcal{A}$ is closed in $\mathcal{X}_l$ with dense domain $\mathcal{Y}_l$, and since the embedding $\mathcal{Y}_l \subset \mathcal{X}_l$ is compact, $\mathcal{A}$ has a compact resolvent. Therefore, the spectrum of $\mathcal{A}$ consists only of discrete eigenvalues with no finite point of accumilation.

From (A.4) it follows that the nonlinear term $\mathcal{F}$ in (A.3) is smooth as a map $\mathcal{F} : \mathbb{R} \times \mathcal{X}_l \to \mathcal{X}_l$. Then the version of the centre manifold reduction given in [12] can be applied, provided the following lemma holds.

**Lemma A.2.** *Denote by $\mathcal{W}$ the closure in $\mathcal{X}_l$ of the range of $\mathcal{F}$. Then there exist a constant $C > 0$ and $\widehat{q} > 0$ such, that for any $q \in \mathbb{R}$, $|q| > \widehat{q}$ the following inequality holds:*

$$||(\mathcal{A} - iq)^{-1}||_{\mathcal{W} \to \mathcal{X}_l} \leq \frac{C}{|q|}.$$

The proof of this lemma is completely analogous to that one of lemma 4.1 in [5], and we omit it here.

Then we can apply the centre manifold reduction theorem, and obtain that all small bounded solutions of (A.3) are of the form

$$\mathbf{w} = a_0\varphi_0 + a_1\varphi_1 + \Phi(\mu, a_0, a_1), \tag{A.7}$$

where $\varphi_0$, $\varphi_1$ are the generalized eigenvectors of $\mathcal{A}$, associated to the double zero critical eigenvalue: $\mathcal{A}\varphi_0 = 0$, $\mathcal{A}\varphi_1 = \varphi_0$ and $\Phi$ is a nonlinear function of its arguments. The amplitudes $a_0$ and $a_1$ satisfy the reduced system:

$$\dot{a}_0 = a_1, \quad \dot{a}_1 = f(\mu, a_0, a_1), \tag{A.8}$$

where $f$ is a nonlinear function of its arguments.

The system (A.8) can be put in normal form

$$\begin{aligned} \dot{a}_0 &= a_1 \\ \dot{a}_1 &= -k^2\mu a_0 + \beta a_0^2 + \gamma a_0^3 + O(\mu a_0, a_0^2), \end{aligned} \tag{A.9}$$

where the coefficients $\beta$ and $\gamma$ are found to be

$$\beta = 0, \quad \gamma = \frac{\Delta}{4}, \quad \Delta = \frac{1}{6(1-5k^2)} < 0.$$

The localized solution of (A.9) is then given by

$$\widehat{a}_0 = \pm k\sqrt{\frac{2\mu}{\gamma}} \operatorname{sech} k\sqrt{|\mu|}y + 0(\mu). \tag{A.10}$$

From (A.7), coming back to the old notation for the spatial variable, one gets the expression for the surface deviation $\eta$:

$$\eta = \pm k\sqrt{\frac{2\mu}{\gamma}} \operatorname{sech} k\sqrt{|\mu|}y \cos(kx - V_1 t) + 0(\mu). \tag{A.11}$$

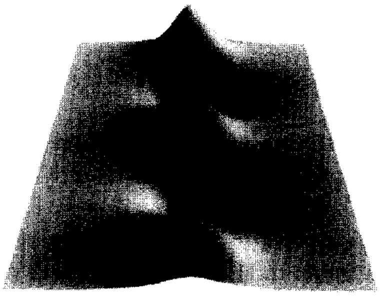

FIGURE 3. Shape of the surface wave of elevation.

The shape of the wave of elevation is presented in Fig.3. This wave is a subcritical one, i.e. its speed $V$ is less than the critical speed $V_1$ ($\mu < 0$).

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